Online Appendix for "Approximate Resolution of Stochastic Choice-based Discrete Planning"

A.1. Proofs

A.1.1. Proof of Proposition 1

Given a feasible $\bar{\boldsymbol{x}} \in \Omega$, define $\mathcal{J}^o = \{j \in \mathcal{J} : \bar{x}_j = 1\}$ and $j_i^* = \arg\max_{j' \in \mathcal{J}^o} u_{ij'}$. An optimal primal solution $\bar{y}$ to **[SP$_i$]** can be easily obtained as

$$\bar{y}_{ij} = \begin{cases} 1, & j = j_i^* \\ 0, & \text{otherwise} \end{cases} \tag{A.1}$$

Now, given $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$, the reduced KKT conditions without primal feasibility are

$$\bar{\nu}_{ij}(\bar{x}_j - \bar{y}_{ij}) = 0, \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \tag{A.2}$$

$$\bar{\mu}_{ik}(1 - \bar{x}_k - \sum_{j \in \mathcal{J}: u_{ij} < u_{ik}} \bar{y}_{ij}) = 0, \forall i \in \mathcal{I}, \forall k \in \mathcal{J}, \tag{A.3}$$

$$r_{ij} - \bar{\lambda}_i - \bar{\nu}_{ij} - \sum_{k \in \mathcal{J}: u_{ik} > u_{ij}} \bar{\mu}_{ik} \leq 0, \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \tag{A.4}$$

$$\bar{\boldsymbol{\nu}}_i, \bar{\boldsymbol{\mu}}_i \geq 0. \tag{A.5}$$

When $j \in \mathcal{J}^o$, we have $\bar{\nu}_{ij}(\bar{x}_j - \bar{y}_{ij}) = 0$ since $\bar{\nu}_{ij} = 0$; when $j \in \mathcal{J} \setminus \mathcal{J}^o$, we have $\bar{\nu}_{ij}(\bar{x}_j - \bar{y}_{ij}) = 0$ since $\bar{x}_j - \bar{y}_{ij} = 0$. Therefore, (A.2) is satisfied.

When $k = j_i^*$, we have $\bar{\mu}_{ik}(1 - \bar{x}_k - \sum_{j \in \mathcal{J}: u_{ij} < u_{ik}} \bar{y}_{ij}) = 0$ since $1 - \bar{x}_{j_i^*} - \sum_{j \in \mathcal{J}: u_{ij} < u_{ik}} \bar{y}_{ij} = 1 - 1 - 0 = 0$; when $k \in \mathcal{J} \setminus \{j_i^*\}$, we have $\bar{\mu}_{ik}(1 - \bar{x}_k - \sum_{j \in \mathcal{J}: u_{ij} < u_{ik}} \bar{y}_{ij}) = 0$ since $\bar{\mu}_{ik} = 0$. Therefore, (A.3) is satisfied.

When $j = j_i^*$, we have $r_{ij} - \bar{\lambda}_i - \bar{\nu}_{ij} - \sum_{k \in \mathcal{J}: u_{ik} > u_{ij}} \bar{\mu}_{ik} = r_{ij_i^*} - r_{ij_i^*} - 0 - 0 = 0$; when $j \in \mathcal{J}^o \setminus \{j_i^*\}$, we have $r_{ij} - \bar{\lambda}_i - \bar{\nu}_{ij} - \sum_{k \in \mathcal{J}: u_{ik} > u_{ij}} \bar{\mu}_{ik} = r_{ij} - \bar{\lambda}_i - [\max_{j \in \mathcal{J}^o \setminus \{j_i^*\}} r_{ij} - \bar{\lambda}_i]_+ \leq 0$; when $j \in \mathcal{J} \setminus \mathcal{J}^o$, we have $r_{ij} - \bar{\lambda}_i - \bar{\nu}_{ij} - \sum_{k \in \mathcal{J}: u_{ik} > u_{ij}} \bar{\mu}_{ik} = r_{ij} - \bar{\lambda}_i - \sum_{k \in \mathcal{J}: u_{ik} > u_{ij}} \bar{\mu}_{ik} - [r_{ij} - \bar{\lambda}_i - \sum_{k \in \mathcal{J}: u_{ik} > u_{ij}} \bar{\mu}_{ik}]_+ \leq 0$. Therefore, (A.4) is satisfied.

Finally, we check the dual and primal objectives, i.e., $\bar{\lambda}_i + \sum_{j \in \mathcal{J}} \bar{\nu}_{ij} \bar{x}_j + \sum_{k \in \mathcal{J}} \bar{\mu}_{ik}(1 - \bar{x}_k) = r_{ij_i^*} + \sum_{j \in \mathcal{J}^o} 0 \cdot 1 + \sum_{j \in \mathcal{J} \setminus \mathcal{J}^o} [r_{ij} - \bar{\lambda}_i - \sum_{k \in \mathcal{J}: u_{ik} > u_{ij}} \bar{\mu}_{ik}]_+ \cdot 0 + 0 \cdot (1 - 1) + \sum_{j \in \mathcal{J} \setminus \{j_i^*\}} 0 \cdot (1 - 0) = r_{ij_i^*} = \sum_{j \in \mathcal{J}} r_{ij} \bar{y}_{ij}$. That is, the primal objective is equal to the dual objective. Altogether, we show that the proposed $(\bar{\lambda}_i, \bar{\boldsymbol{\nu}}_i, \bar{\boldsymbol{\mu}}_i)$ is an optimal solution to **[DSP$_i$]**. □

A.1.2. Proof of Proposition 2

To start, Constraint (25c) can be written as

$$y_{ij} \leq x_j, \forall j \in \mathcal{J}, \tag{A.6}$$

$$y_{ij} \le 1-\delta_{ijk}x_k, \forall j \in \mathcal{J}, \forall k \in \mathcal{J}. \tag{A.7}$$

It suffices to show that Constraint (A.7) has the same effect as Constraint (15e). For any $i \in \mathcal{I}, k \in \mathcal{J}$, define set $\mathcal{J}_i^k = \{j \in \mathcal{J} : U_{ij} < U_{ik}\}$. Then Constraint (A.7) is equivalent to $y_{ij} \le 1 - x_k, \forall j \in \mathcal{J}_i^k$. If option k is offered, then we have $x_k = 1$ and $y_{ij} = 0, \forall j \in \mathcal{J}_i^k$, meaning that any option with lower utility than option k will not be chosen; if option k is not offered, then we have $x_k = 0$ and $y_{ij} \le 1, \forall j \in \mathcal{J}_i^k$, which is automatically satisfied. Therefore, Constraint (A.7) has the same effect as Constraint (15e) and **[nSP$_i$]** is equivalent to **[SP$_i$]**. □

A.1.3. Proof of Proposition 3

Under each scenario $i \in \mathcal{I}$, define $\mathcal{J}^c = \left\{j \in \mathcal{J} : \pi_j > \pi_{m_i^*}\right\}$. Then we can construct $\bar{\boldsymbol{y}}$ as

$$\bar{y}_{ij} = \begin{cases} \bar{\beta}_{ij} & ,\forall j \in \mathcal{J}^c, \\ 1 - \sum_{j \in \mathcal{J}^c} \bar{\beta}_{ij} & ,j = j_i^*, \\ 0 & ,\text{otherwise} \end{cases} \tag{A.8}$$

The reduced KKT conditions without primal feasibility are

$$r_{ij} - \bar{\lambda}_i' - \bar{\eta}_{ij} \le 0, \forall j \in J, \tag{A.9}$$

$$\bar{\eta}_{ij}(\bar{\beta}_{ij} - \bar{y}_{ij}) = 0, \forall j \in J, \tag{A.10}$$

$$\bar{\eta}_{ij} \ge 0, \forall j \in J. \tag{A.11}$$

We thus have $\bar{\lambda}_i' + \bar{\eta}_{ij} = r_{i\pi_{m_i^*}} + [r_{ij} - r_{i\pi_{m_i^*}}]_+ \ge r_{ij}, \forall j \in \mathcal{J}$. Therefore, (A.9) is satisfied.

When $j \in \mathcal{J}^c$, we have $\bar{\eta}_{ij}(\bar{\beta}_{ij} - \bar{y}_{ij}) = 0$ since $\bar{\beta}_{ij} - \bar{y}_{ij} = 0$; when $j = \mathcal{J} \setminus \mathcal{J}^c$, we have $\bar{\eta}_{ij}(\bar{\beta}_{ij} - \bar{y}_{ij}) = 0$ since $\bar{\eta}_{ij} = 0$. Therefore, (A.10) is satisfied.

Finally, we check the primal and dual objectives: $\bar{\lambda}_i' + \sum_{j \in \mathcal{J}} \bar{\eta}_{ij}\bar{\beta}_{ij} = \bar{\lambda}_i' + \sum_{j \in \mathcal{J}^c} \bar{\eta}_{ij}\bar{y}_{ij} + \sum_{j \in \mathcal{J}\setminus\mathcal{J}^c} \bar{\eta}_{ij}\bar{\beta}_{ij} = r_{i\pi_{m_i^*}} + \sum_{j \in \mathcal{J}^c}(r_{ij} - r_{i\pi_{m_i^*}})\bar{y}_{ij} = r_{i\pi_{m_i^*}}(1 - \sum_{j \in \mathcal{J}^c} \bar{y}_{ij}) + \sum_{j \in \mathcal{J}^c} r_{ij}\bar{y}_{ij} = \sum_{j \in \mathcal{J}} r_{ij}\bar{y}_{ij}$, which states that dual and primal objectives are equal. Altogether, we prove that the proposed $(\bar{\lambda}_i', \bar{\boldsymbol{\eta}}_i)$ is an optimal solution to **[nDSP$_i$]**. □

A.1.4. Proof of Proposition 4

Let $k_{ij}^* = \arg\min_{k \in \mathcal{J}} 1 - \delta_{ijk}\bar{x}_k$. It directly follows from the definition of $\boldsymbol{\beta}, \bar{\boldsymbol{\nu}}', \bar{\boldsymbol{\mu}}'$ that

$$\theta_i \le \bar{\lambda}_i' + \sum_{j \in \mathcal{J}} \bar{\eta}_{ij}\bar{\beta}_{ij} \tag{A.13}$$

$$= \bar{\lambda}_i' + \sum_{j \in \mathcal{J}} \bar{\eta}_{ij}\left(\mathbb{I}\left\{\bar{\beta}_{ij} = \bar{x}_j\right\} \cdot x_j + \mathbb{I}\left\{\bar{\beta}_{ij} = 1 - \delta_{ijk_{ij}^*}\bar{x}_{k_{ij}^*}\right\} \cdot (1 - \delta_{ijk_{ij}^*}x_{k_{ij}^*})\right) \tag{A.14}$$

$$= \bar{\lambda}_i' + \sum_{j \in \mathcal{J}} \bar{\nu}_{ij}'x_j + \sum_{j \in \mathcal{J}} \bar{\mu}_{ijk_{ij}^*}'(1 - \delta_{ijk_{ij}^*}x_{k_{ij}^*}) \tag{A.15}$$

$$= \bar{\lambda}_i' + \sum_{j \in \mathcal{J}} \bar{\nu}_{ij}'x_j + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}} \bar{\mu}_{ijk}'(1 - \delta_{ijk}x_k) \tag{A.16}$$

$$= \bar{\lambda}_i' + \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}} \bar{\mu}_{ijk}' + \sum_{j \in \mathcal{J}}(\bar{\nu}_{ij}' - \sum_{k \in J} \delta_{ikj}\bar{\mu}_{ikj}')x_j \tag{A.17}$$

Equalities (A.15) and (A.16) comes from the fact that $\bar{\mu}_{ijk}' = 0, \forall k \ne k_{ij}^*, \forall i \in \mathcal{I}, \forall j \in \mathcal{J}$. □

A.2. Solution quality test

This section provides solution quality results on three applications. We also compare Monte Carlo Sampling (MCS) and Latin Hypercube Sampling (LHS).

A.2.1. Results of CAOP under multinomial probit model

We apply our SBBD on the multinomial probit model, which does not have closed-form formulation of the choice probability. Here, the utility of an alternative $j \in \mathcal{J}$ is given by $U_j = V_j + \xi_j$, where V_j a deterministic component and ξ_j follow some i.i.d. normal distribution. We generate the problem instances as follow: both the deterministic utility V_j and reward R_j is generated from a uniform distribution on the interval [0,100] (with $R_0 = 0$). The random utility ξ follows normal distribution with mean 0 and variance 100.

We use the estimated gaps Δ and $\Delta_{0.95}$ to assess the solution quality. Table A.1 reports the results. Obviously, LHS reduces both the variance σ and gaps compared with MCS. Moreover, when $N = 1000$, our SBBD supported by LHS is able to obtain small Δ values and thus proven high-quality solution.

Table A.1 Solution validation for CAOP under multinomial probit model.

N	$\|\mathcal{A}\|$	τ	MCS					LHS				
			$\hat{v}$	$\bar{v}_N^M$	σ	Δ	$\Delta_{0.95}$	$\hat{v}$	$\bar{v}_N^M$	σ	Δ	$\Delta_{0.95}$
300	100	10	82.48	83.88	0.3853	1.70	2.47	82.45	83.39	0.2863	1.14	1.71
500	100	10	82.77	83.83	0.4011	1.28	2.08	82.74	83.16	0.2626	0.51	1.03
1000	100	10	82.95	83.26	0.4966	0.37	1.36	82.83	83.05	0.0943	0.27	0.45
300	200	10	86.57	88.12	0.4224	1.79	2.60	86.83	88.10	0.4275	1.46	2.27
500	200	10	87.03	88.35	0.4407	1.52	2.35	87.14	87.97	0.4138	0.95	1.74
1000	200	10	87.45	88.33	0.5272	1.01	2.00	87.52	87.89	0.3809	0.42	1.14

A.2.2. Results of FLoP and MSMFLP

This section provides complete results of solution quality test for FLoP and MSMFLP. The dataset is the same as Section 5 in the main text. Tables A.2 and A.3 report the estimated gaps. Obviously, LHS significantly reduces the variance of objective as well as the approximation error compared with MCS. Moreover, for LHS, we include the change of Δ and $\Delta_{0.95}$ when problem size (i.e., $|\mathcal{A}|$ and $|\mathcal{J}|$) increases in the parenthesis. For example, given $N = 300$ and $\tau = 5$, Δ increases from 0.83% to 1.79% when $|\mathcal{A}|$ increases from 10 to 20. Therefore, we put $1.79 - 0.83 = 0.96$ in the parenthesis. Overall, we observe only slight increase of Δ and $\Delta_{0.95}$ for both FLoP (less than 1%) and MSMFLP (less than 0.4%).

A.3. Efficiency test on CAOP under multinomial probit model

This section provides additional efficiency test results on CAOP under multinomial probit model. The utility of an alternative $a \in \mathcal{A}$ is $U_a = V_a + \xi_a$. Both V_a and R_a are generated from $\mathcal{U}[0, 100], \forall a \in \mathcal{A}$. We then set $V_0 = 50$ and $R_0 = 0$. The random term ξ_j follows i.i.d. normal distribution with mean 0 and variance $\mathcal{V}$. We employ the same stabilization procedure in Stage 1 as the exponomial choice model in Section 5.1 of the main text.

Table A.2 Solution validation for FLoP.

N	$\lvert\mathcal{A}\rvert$	τ	MCS $\hat{v}$	MCS $\bar{v}_N^M$	MCS σ	MCS Δ	MCS $\Delta_{0.95}$	LHS $\hat{v}$	LHS $\bar{v}_N^M$	LHS σ	LHS Δ	LHS $\Delta_{0.95}$
300	10	5	3.79	3.90	1.7146	2.92	3.66	3.79	3.82	0.5632	0.83	1.08
500	10	5	3.79	3.86	1.2640	1.94	2.49	3.79	3.81	0.4640	0.67	0.87
1000	10	5	3.79	3.82	0.7711	0.77	1.11	3.79	3.80	0.3453	0.18	0.33
300	20	5	3.91	4.09	1.4531	4.61	5.22	3.93	4.00	0.4413	1.79(+0.96)	1.97(+0.89)
500	20	5	3.93	4.03	1.1287	2.48	2.96	3.93	3.98	0.4116	1.18(+0.51)	1.35(+0.48)
1000	20	5	3.93	3.99	0.7212	1.37	1.68	3.93	3.95	0.3280	0.43(+0.25)	0.57(+0.24)

Table A.3 Solution validation for MSMFLP.

N	$\lvert\mathcal{J}\rvert$	τ	MCS $\hat{v}$	MCS $\bar{v}_N^M$	MCS σ	MCS Δ	MCS $\Delta_{0.95}$	LHS $\hat{v}$	LHS $\bar{v}_N^M$	LHS σ	LHS Δ	LHS $\Delta_{0.95}$
300	100	20	38.18	41.29	0.2214	8.15	9.10	39.26	39.86	0.0854	1.53	1.89
500	100	20	38.83	39.42	0.1636	6.15	6.85	39.42	39.77	0.0513	0.89	1.10
1000	100	20	39.24	40.35	0.1252	2.82	3.35	39.46	39.61	0.0336	0.38	0.52
300	200	20	38.80	42.24	0.2424	8.87	9.90	40.12	40.85	0.0892	1.82(+0.29)	2.19(+0.30)
500	200	20	39.33	41.89	0.1857	6.51	7.29	40.19	40.63	0.0566	1.10(+0.21)	1.33(+0.23)
1000	200	20	39.76	41.69	0.1184	4.84	5.53	40.23	40.53	0.0327	0.74(+0.36)	0.87(+0.35)

Table A.4 reports the computational result. Apparently, SBBD outperforms BIBC by a large margin. Moreover, compared to MILP, SBBD is faster for all the 24 instances. Particularly, the superiority of SBBD becomes even more significant for the eight "hard" instances with $N = 1000$.

Table A.4 Efficiency test on CAOP under multinomial probit model

N	$\lvert\mathcal{J}\rvert$	τ	$\mathcal{V}$	SBBD t[s]	SBBD #N	SBBD #C	SBBD rgap[%]	SBBD ogap[%]	MILP t[s]	MILP #N	MILP rgap[%]	MILP ogap[%]	BIBC t[s]	BIBC #N	BIBC #C	BIBC rgap[%]	BIBC ogap[%]
300	100	10	100	6.2	85	10666	0.28	0.00	7.4	1	0.00	0.00	142.4	3984	2513	3.80	0.00
			200	4.7	38	8015	0.38	0.00	10.0	1	0.00	0.00	145.1	4329	2184	3.49	0.00
		20	100	5.0	1	9092	0.00	0.00	6.7	1	0.00	0.00	727.3	30937	2266	5.47	0.00
			200	5.0	11	8929	0.07	0.00	6.9	1	0.00	0.00	2411.0	57162	2159	5.14	0.00
	200	10	100	14.4	1266	15554	1.39	0.00	43.1	24	0.92	0.00	498.4	10274	3230	2.92	0.00
			200	10.9	550	11090	1.06	0.00	44.4	49	0.73	0.00	214.3	4921	3126	2.01	0.00
		20	100	10.7	304	10885	0.63	0.00	63.8	9	0.36	0.00	3600.0	39168	3576	3.73	1.45
			200	9.8	171	11357	0.50	0.00	64.9	7	0.22	0.00	3600.0	31076	3694	3.01	1.14
500	100	10	100	10.2	1	23464	0.00	0.00	10.0	1	0.00	0.00	443.6	5196	4564	4.41	0.00
			200	8.3	203	12840	0.70	0.00	11.9	1	0.00	0.00	195.0	4322	3602	2.94	0.00
		20	100	4.7	1	7630	0.00	0.00	10.0	1	0.00	0.00	3600.0	32683	3860	5.79	0.52
			200	7.4	19	10769	0.25	0.00	10.1	1	0.00	0.00	3600.0	23834	3763	5.33	0.98
	200	10	100	21.6	732	22384	1.16	0.00	106.7	36	0.59	0.00	1338.5	7818	4025	3.20	0.00
			200	27.6	1125	24099	1.27	0.00	141.9	29	0.59	0.00	3600.0	18919	4751	3.95	0.87
		20	100	23.0	538	22674	0.54	0.00	131.4	1	0.00	0.00	3600.0	10352	5187	2.39	0.69
			200	22.5	446	22393	0.41	0.00	107.2	15	0.55	0.00	3600.0	9691	4999	3.08	1.58
1000	100	10	100	22.9	124	41533	0.38	0.00	49.1	1	0.00	0.00	1554.1	2742	7234	3.85	0.00
			200	29.4	501	41781	0.81	0.00	44.0	1	0.00	0.00	2001.9	4447	9963	3.76	0.00
		20	100	17.4	1	26118	0.00	0.00	51.4	1	0.00	0.00	3600.0	5284	7544	5.39	2.46
			200	15.2	1	23568	0.00	0.00	52.0	1	0.00	0.00	3600.0	3726	6800	5.36	2.68
	200	10	100	59.5	1285	33294	1.91	0.00	228.3	15	0.89	0.00	3600.0	2982	8859	2.44	0.87
			200	83.7	1515	31773	1.95	0.00	341.8	67	1.21	0.00	3600.0	2855	8539	2.76	1.44
		20	100	52.4	687	50238	0.52	0.00	293.4	18	0.77	0.00	3600.0	2225	9867	3.10	2.62
			200	60.3	537	58418	0.59	0.00	336.5	47	0.98	0.00	3600.0	1649	9902	2.93	2.43